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Aerospace System Guidance and Control
Lesson VI

Navigation

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Abstract

A navigation system is necessary for any kind of vehicle that has to move in a open environment and become more and more necessary the less constraints it has in its dynamics. Take a train for example, it has just one degree of freedom (if we do not include interchanges between rails) therefore it just need to know where it is along the rail, but the position of the rails are known beforehand. On the other hand a plane has six degrees of freedom and it has no physically fixed roads and it needs more instruments to understand its location.

Moreover we must be aware that a physical system is a time continuous process while measurement, since we make use of digital computers and similar devices, are taken as discrete samples of such continuous signals.

One thing that must be clear before using the information from the sensors: no measure is perfect. Measurements are affected by various kind of noises and errors, therefore having knowledge of random processes is quite a must.

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Chapter 1

Principles of Navigation

Navigation is, in general, a simple concept and is easily understood by everyone, however things are not so easy as they seems. To navigate means to know were you are and where you are going and is no simple task. First of all to determine the state (position, velocity, attitude) of a vehicle some measurement are necessary: there is no navigation system that does not uses sensors. Sensors measurement are affected by various kind of noises and often they measures something different from the state and thus a proper elaboration of the data is to be used.

1.1 Determining attitude

To determine the orientation of a body usually requires to determine the position vector of different known location. The good old compass gives the direction of the magnetic pole and that gives, with some errors due to misalignment of magnetic poles and geographic poles, the north direction. If the body is not moving a simple pendulum can give the zenith direction. Modern sensors are used in a similar way, but of course they are more complex.

One thing that should be understood as early as possible is that with just one vector measurement is not possible to determine the full orientation of a body. Some restriction given by the dynamics can be applied and reduce the uncertainty or simply discarding the idea of reconstructing the full orientation of the body and use just one single vector measurement.

There are various kind of methods to determine attitude using vector measurement, some use statistics, other shear computational power, and so on. In general is preferred to have more than three measures in order to reduce the effects of instrument noise, however this often requires the computation of a pseudo-inverse[1]. Another preference is to have the measured vectors as orthogonal as possible for the sake of obtaining a good result.

Then another issue is to determine wich kind of parameters to use for the attitude determination: direction cosines, Euler angles, quaternions, etc. Some techniques are better with one parametrization, others with another.

[1] If a matrix is not square can be "inverted" using one between two formulas and the results is the inverse matrix that minimized the squared errors of the fitting.

1.2 Determining position

Determining position is even more challenging that determining the attitude of the body. Again one of the most used methods is to determine the distance from known locations and similarly determine the actual position. This can be achieved using ground stations, as often used in the aeronautics, or satellites as for GPS. This kind of determination can be precise but needs the installation of other systems, therefore is not always possible to exploit it. A lot of research is carried out to determine position in GPS-denied environment.
Position, as attitude, can be determined by integration of velocities or accelerations, however the noisier the measures the worst the outcome is.
One important thing in position determination is to set a reference frame to use for the measurement, otherwise no meaningful position can be determined. A GPS gives the altitude, latitude and longitude of the body w.r.t. a reference frame fixed with the planet, therefore in many cases a transformation of coordinates has to be done.

Chapter 2

Random Process (basics)

The first thing to mention regarding random processes is to address what is deterministic and what is not. A deterministic process that is measured in this instant can be determined in a future time instant if previous measurement are known. On the other hand a random process cannot be determined even if all measurements are taken from the dawn of time.

Even if a process is non deterministic it is still possible to extract useful information from it. Let us take an ensemble of N samples of a random process, we can define the mean[1] and the variance[2] as follows

$$\mu_x = \sum_{i=1}^{N} x_i$$
$$V_x = \sigma_x^2 = \frac{1}{N}\sum_{i=1}^{N}\left[x_i - \mu_x\right]^2$$

If we think about a signal that is continuous in time, we must take another step and consider the time dependencies

$$\mu_x = \lim_{T\to\infty}\left[\frac{1}{T}\int_{t'-T}^{t'} x(\tau)\, d\tau\right]$$
$$V_x = \sigma_x^2 = \lim_{T\to\infty}\left[\frac{1}{T}\int_{t'-T}^{t'} \left[x(\tau) - \mu_x\right]^2 d\tau\right]$$

however this implies that all measurements from the beginning must be known. This is not the case in all real applications, therefore we can define estimations of the mean and variance as

[1] Mean value, the most probable value a signal can take
[2] Statistical measure of data dispersion, it is non-negative.

$$\hat{\mu}_x = \frac{1}{T} \int_{t'-T}^{t'} x(\tau)\, d\tau$$

$$\hat{V}_x = \hat{\sigma}_x^2 = \frac{1}{T} \int_{t'-T}^{t'} [x(\tau) - \hat{\mu}_x]^2\, d\tau$$

that represent the average and variance taken on a finite time window of length T.
Another useful quantity is the Probability Density Function $p_x(\alpha)$ that represent the probability that x lies in the interval $\alpha \le x \le \alpha + d\alpha$. It can be seen as a statistical descriptor of the wave form of $x(t)$. The PDF can be used to determine other useful parameters and insights, however this is not the scope of the course. We will concentrate on the implication the knowledge of the PDF can provide us.
An operator that will be useful later on is the Expected Value

$$E[x_k(t_1)] = \lim_{n_s \to \infty} \left[\frac{1}{n_s} \sum_{k=1}^{n_s} x_k(t_1) \right]$$

that, being a limit, can only be estimated. If the PDF of $x_k(t_1)$ is known it is possible to determine analytically the expected value of $x_k(t_1)$. The PDF can be divided into three categories regarding the way they are determined:

- empirical estimates without theoretical basis,
- rigorous analytical derivation from a physical model,
- hypothetical functions that seems close to reality.

The expectation is equivalent to the first moment of $p_{x_{t_1}}$ with $\alpha = 0$ such that

$$E[x_{t_1}] = \int_{-\infty}^{+\infty} \alpha p_{x_{t_1}}(\alpha)\, d\alpha$$

The expected value has different properties depending on the signal it analyze. The expected value is a linear operator, in fact

$$\begin{aligned} E[x] &= \mu_x \\ E[x+y] &= \mu_x + \mu_y \\ E[ax] &= a\mu_x, \qquad a \exists \mathbb{R} \\ E\left[(x+y)^2\right] &= E\left[x^2\right] + 2E[xy] + E\left[y^2\right] \end{aligned}$$

Given two signals x and y if we have $E[xy] = 0$ we have that the two processes are orthogonal. If these process are independent it is sufficient for μ_x or μ_y to be zero. Orthogonality do not implies independence. Moreover, if the processes are orthogonal we have

$$E\left[(x+y)^2\right] = E\left[x^2\right] + E\left[y^2\right]$$

Chapter 3

Lynear Systems : Observer

Let us take a linear MIMO system that represent the behavior of a real system as follows

$$\begin{cases} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{cases}$$

where we have multiple measures $\mathbf{y}$. What we want to build is a linear system that estimates the state $\mathbf{x}$ using the measures $\mathbf{y}$: this is the so called "Observer". The observer has to be a dynamical system, otherwise it could not follow the evolution of the system state through time.

3.1 Observability

The first thing to do is to answer the question: is the system is observable? This means that we must know if the measures that we are taking are enough to estimate the whole state. The classical method is to determine the rank of the observability matrix defined as follows:

$$\mathbf{O}_{m \cdot n \times n} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}$$

The observability matrix has $m \cdot n$ rows where n is the number of states and m is the number of measures. The maximum rank that this matrix can assume is n, therefore if the rank is equal to the number of states the whole system is observable, otherwise we cannot estimate the whole state from such measures. This is a quick method to understand if the equipment we have is enough or we must add other sensors.

3.2 Observer derivation

The observer is a linear system such as

$$\dot{\hat{\mathbf{x}}} = \hat{\mathbf{A}}\hat{\mathbf{x}} + \hat{\mathbf{B}}\mathbf{u} + \mathbf{L}\mathbf{y}$$

we define the error as

$$\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$$

and by subtraction we get the error evolution in time

$$
\begin{aligned}
\dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}} &= \mathbf{A}\mathbf{x} - \hat{\mathbf{A}}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} - \hat{\mathbf{B}}\mathbf{u} - \mathbf{L}\mathbf{y} \\
\dot{\mathbf{e}} &= \mathbf{A}\mathbf{x} - \hat{\mathbf{A}}\left(\mathbf{x} - \mathbf{e}\right) + \mathbf{B}\mathbf{u} - \hat{\mathbf{B}}\mathbf{u} - \mathbf{L}\mathbf{C}\mathbf{x} \\
\dot{\mathbf{e}} &= \hat{\mathbf{A}}\mathbf{e} + \left(\mathbf{A} - \hat{\mathbf{A}} - \mathbf{L}\mathbf{C}\right)\mathbf{x} + \left(\mathbf{B} - \hat{\mathbf{B}}\right)\mathbf{u}
\end{aligned}
$$

Again we have a linear system that describe the evolution in time of the state determination error. If this system is stable we can affirm that, under the linearity assumption, the error does not increase over time. This case is, however, slightly more complex than usual. As we can see the error evolution depends also on external factors. We do not want to have it dependent on the state or control values,therefore we impose the following constraints

$$
\begin{cases}
\mathbf{0} = & \mathbf{A} - \hat{\mathbf{A}} - \mathbf{L}\mathbf{C} \\
\mathbf{0} = & \mathbf{B} - \hat{\mathbf{B}}
\end{cases}
$$

This means that the magnitude of the state or of the control action will not affect the error evolution in time. This is done to prevent error enhancement. The result give us the values of the observer matrices

$$
\begin{cases}
\hat{\mathbf{A}} = & \mathbf{A} - \mathbf{L}\mathbf{C} \\
\hat{\mathbf{B}} = & \mathbf{B}
\end{cases}
$$

This shows us the design variable that we can shape accordingly to our needs: the observer gain matrix $\mathbf{L}$. We can choose $\mathbf{L}$ such that the dynamics of the error is pre-determined. In fact $\mathbf{L}$ can shift the eigenvalue of $\hat{\mathbf{A}}$ to have all negative real part, making the error dynamics stable and confined. This technique is often called pole placement method.

3.3 Kalman Filter

As we have seen before no real process is deterministic, therefore we usually have to cope with a system like this one

$$
\begin{cases}
\dot{\mathbf{x}} & = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{F}\mathbf{n} \\
\mathbf{y} & = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} + \mathbf{v}
\end{cases}
$$

where $\mathbf{n}$ is the process noise and $\mathbf{v}$ the measurement noise. These noises are assumed to be white noise with zero mean and a Gaussian distribution. We can define the covariance of these noises as

$$
\begin{cases}
\mathbf{Q} & = \mathrm{E}\left[\mathbf{n}\mathbf{n}^{\mathrm{T}}\right] \\
\mathbf{R} & = \mathrm{E}\left[\mathbf{v}\mathbf{v}^{\mathrm{T}}\right]
\end{cases}
$$

the goal is to find an observer that optimally weights the measure and the model uncertainties and produce the best estimate of the state.

$$
\begin{aligned}
\dot{\hat{\mathbf{x}}} &= \left(\mathbf{A} - \mathbf{L}\mathbf{C}\right)\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}\mathbf{v} \\
\dot{\mathbf{e}} &= \left(\mathbf{A} - \mathbf{L}\mathbf{C}\right)\mathbf{e} + \mathbf{F}\mathbf{n} + \mathbf{L}\mathbf{v}
\end{aligned}
$$

To do so we use the optimal control theory and simply minimize the quadratic error of the estimate. In practice what we need is to have:

- a linear model of the system
- the noises covariance

- an initial state estimate error covariance $\mathbf{P}_0$[1]

We can determine the evolution in time of the state error covariance $\mathbf{P}$ via Riccati differential equation, however if we assume that the system has passed its transitory region and it is stationary, we can obtain an algebraic solution. The derivation of all these results is skipped for the sake of brevity and since this is not the goal of the course.

$$\mathbf{L} = \mathbf{P}_\infty \mathbf{C}^\mathsf{T} \mathbf{R}^{-1}$$
$$\mathbf{P}_\infty (\mathbf{A} - \mathbf{LC})^\mathsf{T} + (\mathbf{A} - \mathbf{LC}) \mathbf{P}_\infty + \mathbf{FQF}^\mathsf{T} + \mathbf{L}^\mathsf{T} \mathbf{RL} = \mathbf{0}$$
$$\mathbf{P}_\infty \mathbf{A}^\mathsf{T} + \mathbf{A}\mathbf{P}_\infty - 2\mathbf{P}_\infty \mathbf{C}^\mathsf{T} \mathbf{R}^{-\mathsf{T}} \mathbf{C}\mathbf{P}_\infty + \mathbf{FQF}^\mathsf{T} + \mathbf{R}^{-\mathsf{T}} \mathbf{C}\mathbf{P}_\infty \mathbf{R}\mathbf{P}_\infty \mathbf{C}^\mathsf{T} \mathbf{R}^{-1} = \mathbf{0}$$

These are the expressions that gives the so called Linear constant Kalman Filter. When dealing with such family of filters we will use $\mathbf{K}$ instead of $\mathbf{L}$ to address the observer problem. Kalman filtering is a major subject in signal analysis and also in navigation problems.

[1] Notice that all covariance must be symmetric and positive definite

Chapter 4

Kalman Filtering

Kalman filtering has been widely used all around the world in the last decades and it is a method well studied in many fields for many different applications. It can be adapted to many problems and can assume many shapes: the number of KF variant is enormous. We will briefly explain some of the most common and useful applications.

4.1 Recursive Kalman Filter

We have derived before the equations of the constant Kalman gain $\mathbf{K}$ using the stationary Riccati equation, however there is a simple procedure that can be used in case of time dependent systems. In the Riccati equation the solution depends on $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{A}$, if they change in time the solution is not optimal anymore and not correlated to the problem. Therefore there is a way to solve iteratively the Riccati equation and address the problem of such quantities changing in time. The procedure is the following:

1. Predict the state $\mathbf{x}_{k|k-1}$ at the following step using the previous known state estimation. This step can also be solved using a non linear estimation of the state, if this produce better results.

2. Predict the measurement $\tilde{\mathbf{y}}_k$ given the measurement model of the system.

3. Predict the state error covariance $\mathbf{P}$ using the following simplified Riccati equation, the state matrix with the previous condition Φ_{k-1} and the model error covariance $\mathbf{Q}_k$

$$\mathbf{P}_{k|k+1} = \Phi_{k-1}^{T}\mathbf{P}_{k-1|k-1}\Phi_{k-1} + \mathbf{Q}_k$$

4. Compute the Kalman Gain

$$\mathbf{K}_k = \mathbf{P}_{k|k+1}\mathbf{C}_{k-1}^{T}\left(\mathbf{C}_{k-1}\mathbf{P}_{k|k+1}\mathbf{C}_{k-1}^{T} + \mathbf{R}_k\right)^{-1}$$

5. Update the state estimation using the residual $\tilde{\mathbf{y}}_k - \mathbf{y}_k$ where $\mathbf{y}_k$ are the measurement the observer takes as input.

$$\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{K}_k\left(\tilde{\mathbf{y}}_k - \mathbf{y}_k\right)$$

6. Update the state error covariance

$$\mathbf{P}_{k|k} = \left(\mathbf{I} - \mathbf{C}_k^T \mathbf{K}\right) \mathbf{P}_{k|k+1}$$

This procedure can be applied to continuous, discrete and continuous-discrete systems. In the latter case, where the system is continuous and the measurements are discrete, it is needed to just add one step with the sampling of the state and state error estimation covariance matrix.

4.2 Extended Kalman Filter

In case the system is not linear it is possible to use the so called Extended Kalman Filter (EKF) with the iterative procedure described above and linearizing the system at each step around the previous known estimate. The optimality of the KF is lost but this technique has been widely used and its robust. Sometimes it suffers from filter divergence.

$$\begin{cases} \dot{\mathbf{x}} & = a\left(\mathbf{x},\, t\right) + b\left(\mathbf{u},\, t\right) \\ \mathbf{y} & = c\left(\mathbf{x},\, t\right) + d\left(\mathbf{u},\, t\right) \end{cases}$$

In this case a, b, c, d are non linear function. It is possible to return to the previous kind of representation by simply linearizing the functions:

$$\mathbf{A} = \left.\frac{\partial a\left(\mathbf{x},\, t\right)}{\partial \mathbf{x}}\right|_{\mathbf{x}(t)}$$

The matrix $\mathbf{A}$ in general depends on the state and or time, therefore it is often necessary to compute it again. If we talk about discrete systems the process refers to the previous step. In this case the optimality of the Kalman filter is lost, however good results can still be achieved.
In the recursive KF the estimation of the next state and measurements can be carried out using the non-linear model in order to increase precision.

4.3 Optimal Control

The theory behind optimal control is not trivial and the interested readers are suggested to look at ordinary textbooks. One interesting thing to address is that in linear systems the observer problem and the controller problem are disjointed, therefore we can address the control and the observer separately. A controller that makes use of optimal control is often called Linear Quadratic Regulator (LQR) and is, by all means, a linear system that can be written in state space form. If the controller incorporates an optimal estimator (a Kalman Filter) the result is the same and it takes the name Linear Quadratic Gaussian Regulator, due to the fact that the state and measurement error are modeled as white noises with Gaussian distribution. To generate a constant LQGR we need information on the model and measurement error covariance as well as the performance that we want to minimize and the control action that we want to use. Setting all these parameter is not trivial and requires experience, knowledge and extensive trial/error practice.

Chapter 5

Blimp Navigation

In this section we will address some of the possible methods useful for the blimp navigation. First of all we must review the sensors , then we will address the design of navigation using such sensors.

5.1 Sensors

The available sensors are:

- accelerometers,
- gyroscopes,
- magnetometers,
- altimeters,
- ultrasonic ranging sensor.

Some instruments cannot be used alone, other require special handling to give useful information. Some measure something related to the external environment, others make no use of such information.
In general accelerometers and gyroscopes are used together in the so called Inertial Measurement Unit (IMU) and don't require any external information. They prove to be useful when other kind of measures are not possible or suffer from signal losses. On the other hand using an IMU to reconstruct the trajectory is quite difficult and the instrument errors make the position determination error diverge quite easily. accelerometers and gyroscopes are also prone to accumulate bias over time, therefore the accuracy of these instruments is limited in time.
On Earth IMU platform is often used a magnetometer in order to reduce the drift of gyroscope integration: magnetometers can easily give a constant reference direction.
Altimeters measures the external pressure and using an atmospheric model such as the ISA it is possible to compute the altitude. This sensors requires calibration and heavily depends on external conditions, however is able to give precise measurement, even if sometimes may suffer from lag.
The ultrasonic ranging sensor measures the distance of an object giving a signal and measuring the time difference of the reflected signal. The performance of the sensor may vary but its nature is quite different from the others: it has a natural delay.

5.2 Choice of sensors

Picking up all the sensors and putting them on the vehicle is not always a good solution and often is not really possible. In general one can think that the more sensors are available the more precise the estimation will be, however is not always the case. Moreover in general the navigation unit, as other subsystems, have constraints of space, weight and power consumption, therefore the system must be optimized for the task it must fulfill.
If space and weight limits are fulfilled more sensors can be used, however using all at once may not be the wisest solution. It's important to understand that in some conditions certain sensors are useless or can even worsen the estimation. Sometimes certain sensors can be manipulated in a completely different way in order to estimate something different: an accelerometer is used to measure the trajectory, however if the blimp is anchored it can provide the direction of the gravity field and together with the magnetometer can define a good reference frame for the motion.
It is very useful, in the early stage of the design, to define different navigation modes, connected to the different modes of the vehicle during its mission. For example there could be a mode for the attitude determination before moving, a safety check for obstacles using the ranging sensor or a navigation mode for the final approach to target.

5.3 Inertial Navigation System

IMU can be divided into two category: strap down systems or gimballed systems. In the first application the IMU is fixed on the vehicle while on the second application it is mounted on a moving platform that tries to match the accelerometer orientation to that of the inertial systems chosen for motion reference. We will refer only to the first type for structural reasons. since the platform is fixed the passage from body axes to inertial axes must be carried out by the navigation algorithm.
Accelerometers measures a combination of acceleration such that they measure 1g of acceleration skyward while on the ground. In that case they measure the normal force that the ground uses to oppose the gravity force so that the object is not falling. We can therefore say that the acceleration in the inertial frame $\mathbf{a}_i$ are

$$\mathbf{a}_i = \mathbf{R}_{ib}\mathbf{a}_m + \mathbf{g}$$

where $\mathbf{R}_{ib}$ is the rotation matrix that passes a vector from the body frame to the inertial frame, $\mathbf{a}_m$ is the vector of the measured accelerations and $\mathbf{g}$ is the gravity acceleration written in the inertial reference frame. The acceleration in the inertial frame is then integrated twice to get first velocity and then position. During each integration step also the instrument noise is integrated and generates a huge drift in the position.
The variation in time of $\mathbf{R}_{ib}$ is made using the measured angular velocities and a suitable attitude kinematic method as those seen on previous lessons.
One popular approach to refine IMU measurement is to use a indirect Kalman filter where the Inertial Navigation System is the system and another set of sensors provide measures of the INS error. To generate such model one must take into account the noise and the random walk[1] of the IMU. The system thus generated is linear but time dependent. If this kind of approach is pursued one must choose other sensors to be able to measure position and attitude and make sure that the system is observable.

[1] A low frequency error that can be modeled as the integral of a white noise

5.4 Altimeter

The altimeter measures the pressure (barometer) and through a air model gives back the altitude. There are different air models and difference reference pressure, but for our application we will use a basic model. It is important to notice that the measure of altitude is indirect and affected by the air near the vehicle, therefore is not always accurate.

5.5 Ranging Sensor

The ranging sensor output is the distance between the sensor and the first obstacle that crosses its path. In general the sensor should give the distance along the normal direction and it is limited in range. Usually the precision of the sensor depends on the distance from the obstacle, meaning that the nearer it is, the better the measure is. Of course there is a minimum distance that the sensor can measure: if the object is closer the response of the sensor will be anyway the minimum measurable distance.
This ranging sensor can be used to detect obstacle in the blimp path or to determine the distance from the ground to better estimate altitude.
More advanced ranging sensors can be used also to determine the speed of the vehicle through Doppler effect.

Appendix

Function name	Description	Notes
pinv	computes the pseudo inverse	
place	get $\mathbf{L}$ to have certain poles	use $\mathbf{C}^T$
Kalman	Kalman observer design	
lqr	design a LQR controller	
lqgr	design a LQGR controller	
ss	generate a state space object	
obsv	get observability matrix	
ctrb	get controllability matrix	
eig	get eigenvalue of a matrix	
poles	compute poles of a ss system	
pzmap	plot poles and zeros of ss system	

Type "help namefunction" to have the description of input/output and options of the selected function.