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Aerospace System Guidance and Control
Lesson III

System State, Transfer Function and Frequency Response

POLITECNICO DI MILANO

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Abstract

The first step in the control design process is to develop appropriate mathematical models of the system through physical laws or experimental data. In this section, we introduce the state-space and transfer function representations of dynamic systems. Once appropriate mathematical models of a system have been obtained, either in state-space or transfer function form, we may then analyze these models to predict how the system will respond in both time and frequency domains. To put this in context, control systems are often designed to improve stability, speed of response, steady-state error, or prevent oscillations. In this section, we will show how to determine these dynamic properties from the system models.

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Contents

1 State of a Dynamic System **1**
1.1 The State Equation 2
1.2 Output Equations. 3
1.3 State Equation Based Modeling Procedure. 4

2 Transformation From State-Space Equations to Classical Form **5**

3 Transformation from Classical Form to State-Space Representation **10**

4 The Matrix Transfer Function **12**

5 Frequency Response **13**
5.1 The Concept of Frequency Response 13
5.2 Analytical Expressions for the Frequency Response. 14

Chapter 1

State of a Dynamic System

The concept of the state of a dynamic system refers to a minimum set of variables, known as state variables, that fully describe the system and its response to any given set of inputs. In particular a state-determined system model has the characteristic that:
<<*A mathematical description of the system in terms of minimum set of variable* $x_i(t), i = 1...n$ *together with knowledge of those variables at an initial time* $t > t_0$ *are sufficient to predict the future system state and outputs for all time*$t > t_0$ *. This definition asserts that the dynamic behavior of state-determined system is completely characterized by the response of the set of n variables* $x_i(t)$*, where the number n is defined order of the system.*>>
If the system is state-determined, knowledge of its state variables $(x_1(t_0), x_2(t_0), ..., x_n(t_0))$ at some initial time t_0, and the inputs $u_1(t)$ and $u_2(t)$ for $t > t_0$ is sufficient to determine all future behavior of the system. The state variables are an internal description of the system which completely characterize the system state at any time t, and from which any output variables $y_i(t)$ may be computed. System models constructed with the pure and ideal *(linear)* one-port elements (such as mass, spring and damper elements) are state-determined system models. For such systems the number of state variables, n, is equal to the number of independent energy storage elements in the system. The values of the state variables at any time t specify the energy of each energy storage element within the system and therefore the total system energy, and the time derivatives of the state variables determine the rate of change of the system energy. Furthermore, the values of the system state variables at any time t provide sufficient information to determine the values of all other variables in the system at that time.

1.1 The State Equation

A standard form for the state equations is used throughout system dynamics. In the standard form the mathematical description of the system is expressed as a set of n coupled first-order ordinary differential equations, known as the state equations, in which the time derivative of each state variable is expressed in terms of the state variables $x_1(t_0), x_2(t_0), ..., x_n(t_0)$ and system inputs $u_1(t), u_2(t), ...u_r(t)$. In the general case the form of the n is:

$$\begin{aligned} \dot{x_1} &= f_1(\mathbf{x},\mathbf{u},t) \\ \dot{x_2} &= f_2(\mathbf{x},\mathbf{u},t) \\ ... &= ... \\ \dot{x_n} &= f_n(\mathbf{x},\mathbf{u},t) \end{aligned} \tag{1.1}$$

where $\dot{x} = {}^{dx_i}/_{dt}$ and each of the function $f_i(\mathbf{x},\mathbf{u},t), i = 1...n$ may be a general non linear, time varynig function of the state variables, the system inputs, and time.

It is common to express the state equations in a vector form, in which the set of n state variables is written as a *state vector* $\mathbf{x}(t) = [x_1(t), x_2(t), ..., x_n(t)]^T$, and the set of r inputs is written as and input vector $\mathbf{u}(t) = [u_1(t), u_2(t), ..., u_r(t)]^T$. Each state variable is a time varying component of the column vector $\mathbf{x}(t)$.

This form of the state equations explicity represents the basic elements contained in the definition of the state determined system. Given a set of initial conditions (the value of the x_i at some time t_0) and the inputs for $t > t_0$, the state equations explicity specify the derivatives of all state variables. The value of each state variable at some time $\triangle t$ later may then be found by direct integration.

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The system state at any instant may be interpreted as a point in an *n*-dimensional *state space*, and the dynamic state response $\mathbf{x}(t)$, can be interpreted as a path or trajectory traced out in the state space.

In vector notation, the set of n equation in Eqs. (1) may be written:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t) \tag{1.2}$$

where $\mathbf{f}(\mathbf{x}, \mathbf{u}, t)$ is a *vector* function whit *n* components $f_i(\mathbf{x}, \mathbf{u}, t)$.

In this note, we restrict attention primarly to a description of systems that are *linear* and *time-variant* (LTI), that is systems described by linear differential equations with cosant coefficients. For an LTI systems described by linear differential equations with costant coefficients. For an LTI system of order *n*, and with *r* inputs, Eqs. (1) become a set of n coupled firs-order linear differential equations with costant coefficients:

$$\begin{cases} \dot{x}_1 = & a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n + b_{11}u_1 + ... + b_{1r}u_r \\ \dot{x}_2 = & a_{21}x_1 + a_{22}x_2 + ... + a_{:2n}\ x_n + b_{21}u_1 + ... + b_{2r}u_r \\ \vdots & \vdots \\ \dot{x}_n = & a_{n1}x_1 + a_{n2}x_2 + ... + a_{nm}x_n + b_{n1}u_1 + ... + b_{nr}u_r \end{cases} \tag{1.3}$$

where the coefficient e a_{ij} and b_{ij} are constants that describe the system. This set of n equation define the deruvatuves of the state variables to be a weighted sum of the state variables and the system inputs.

Equation (3) may be written compactly in a matrix form:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1r} \\ b_{21} & & b_{2r} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nr} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix} \tag{1.4}$$

which may be summarized as:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{1.5}$$

where the state vector **x** is a column vector of lenght n, the input **u** is a column vector of lenght r, **A** is an $n \times n$ square matrix of the constant coefficients a_{ij}, and **B** is an $n \times r$ matrix of the coefficients b_{ij} that weight inputs.

1.2 Output Equations.

A system *output* is defined to be any sysstem variable of interest. A description of a physical system i terms of a set of state variables does not necessarily include all of thevariables with an immediate (engineering) interest. An important property of the linear state equation description is that all system variables may be represented by a linear combination of the state variables x_i and the system inputs u_i. An arbitrary output variable in a system of order n with r inputs may be written:

$$y(t) = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n + d_1 u_1 + \ldots + d_r u_r \tag{1.6}$$

where the c_i and d_i are constants. If a total of m system variables are defined as outputs, the m such equations may be written as:

$$\begin{aligned} y_1 &= c_{11}x_1 + c_2 x_2 + \ldots + c_{1n}x_n + d_{11}u_1 + \ldots + d_{1r}u_r \\ y_2 &= c_{21}x_1 + c_{22}x_2 + \ldots + c_{2n}x_n + d_{21}u_1 + \ldots + d_{2r}u_r \\ &\vdots \\ y_m &= c_{m1}x_1 + c_{m2}x_2 + \ldots + c_{nm}x_n + d_{m1}u_1 + \ldots + d_{mr}u_r \end{aligned} \tag{1.7}$$

or in a matrix form:

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \ldots & c_{1n} \\ c_{21} & c_{22} & \ldots & c_{2n} \\ \vdots & & & \vdots \\ c_{m1} & c_{m2} & \ldots & c_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} d_{11} & \ldots & d_{1r} \\ d_{21} & & d_{2r} \\ \vdots & & \vdots \\ d_{m1} & \ldots & d_{mr} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix} \tag{1.8}$$

The output equations, Eqs. (8), are commoly written in the compact form:

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \tag{1.9}$$

where **y** is a column vector of the output variables $y_i(t)$, **C** is an $m \times n$ matrix of the costant coefficients c_{ij}that weight the state variables, and **D** is an $m \times r$ matrix of the costant coefficients d_{ij} that weight the system inputs. For many physical system the matrix **D** is the null matrix, ant the output equation reduces to a simple weighted combination of the state variables:

$$\mathbf{y} = \mathbf{C}\mathbf{x} \tag{1.10}$$

1.3 State Equation Based Modeling Procedure.

The complete system model for a linear time invariant sustem consists of: *i)* a set of n state equations, defined in terms of the matrix **A** and **B**, and *ii)* a set of output equations that relate any output variables of interest to the state variables and inputs, and expressed in terms of the **C** and **D** matrices. The task of modeling the system it to derive the elements of the matrices, and to write the system model in the form:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{Ax} + \mathbf{Bu} \\ \mathbf{y} &= \mathbf{Cx} + \mathbf{Du} \end{aligned} \tag{1.11}$$

The matrices **A** and**B** are properties of the system and are determined by the system structure and elements. The ouput equation matrices **C** and **D** are determined by the particular choice of output variables.
The overall modeling procedure developed in this chapter is based on the following steps:

1. determination of the system order n and selection of a set of state variables from the linear graph system representation;

2. generation of a set of state equations and the system **A** and**B**matrices using a well defined methodology. This step is also based on the linear system description;

3. determination of a suitable set of output equations and derivation of the appropriate **C** and **D** matrices.

Chapter 2

Transformation From State-Space Equations to Classical Form

The transfer function and the classical input-output differential equation for any system variable may be found directly from a state space representation through the Laplace transform.
For example, we can find the transfer function and a single first-order differential equation relating the output $y(t)$ to the input $u(t)$ for a system described by the first-order linear state and output equations:

$$\begin{array}{rcrcl} \frac{dx}{dt} & = & ax(t) & + & bu(t) \\ y(t) & = & cx(t) & + & du(t) \end{array} \tag{2.1}$$

Thanks to Laplace transform, we can write the following expression:

$$sX(s) = aX(s) + bU(s) \tag{2.2}$$

which may be rewritten with the state variable $X(s)$ ont the left-hand side:

$$(s-a)X(s) = bU(s). \tag{2.3}$$

Then dividing by $(s-a)$, solve for the state variable:

$$X(s) = \frac{b}{s-a}U(s) \tag{2.4}$$

and substitute into the Laplace transform of the output $Y(s) = cX(s) + dU(s)$:

$$Y(s) = \left[\frac{bc}{s-a} + d\right]U(s) = \frac{ds + (bc - ad)}{(s-a)} \tag{2.5}$$

The transfer function is:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{ds + (bc - ad)}{(s-a)} \tag{2.6}$$

The differential equation is found directly:

$$(s-a)Y(s) = (ds + (bc - ad))U(s) \tag{2.7}$$

and rewriting as a differential equation:

$$\frac{dy}{dy} - ay = d\frac{du}{dt} + (bc - ad)u(t) \tag{2.8}$$

Classical representation of higher-order system may be derived in the same way using the Laplace transform and matrix algebra. A set of linear state and output equations written in standard form:

$$\begin{array}{rcccc} \dot{\mathbf{x}} & = & \mathbf{Ax} & + & \mathbf{Bu} \\ \mathbf{y} & = & \mathbf{Cx} & + & \mathbf{Du} \end{array} \tag{2.9}$$

may be rewritten in the Laplace domain. The system equations are then:

$$\begin{array}{rcccc} s\mathbf{X}(s) & = & \mathbf{A}X(s) & + & \mathbf{B}U(s) \\ \mathbf{Y}(s) & = & \mathbf{C}X(s) & + & \mathbf{D}U(s) \end{array} \tag{2.10}$$

and the state equations may be rewritten:

$$sx(s) - \mathbf{A}x(s) = [s\mathbf{I} - \mathbf{A}]\mathbf{x}(s) = B\mathbf{u}(s) \tag{2.11}$$

where the term $s\mathbf{I}$ creates an $n \times n$ matrix with s on the leading diagonal and zeros elsewhere. (This step is necessary because matrix addition and substraction is only defined for matrices of the same dimension.) The matrix $[s\mathbf{I} - \mathbf{A}]$ which appears frequently throughout linear system theory it is a square $n \times n$ matrix with elements directly related to the $\mathbf{A}$ matrix:

$$[s\mathbf{I} - \mathbf{A}] = \begin{bmatrix} (s - a_{11}) & -a_{12} & \dots & -a_{1n} \\ -a_{21} & (s - a_{22}) & \dots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \dots & (s - a_{nn}) \end{bmatrix} \tag{2.12}$$

The state equations, written in the form of Eq. (22), are a set of n simultaneous operational expressions. The common methods of solving linear algbric equations, for example Gaussion elimination, Cramer's rule, the inverse matrix , elimination and substitution, may be directly applied to linear operational equations such as Eq. (22).

For low-order single-input single-output systems (SISO) the transformation to a classical formulation may be performed in the following steps:

1. take the Laplace transform of the state equations;

2. reorganize each state equation so that all terms in the state variables are on the left-hand side;

3. treat the state equations as a set of simultaneous algebric equations and solve for those state variables required to generate the ouput variable;

4. substitute for the state variables in the output equation;

5. write the output equation in operational form and identify the transfer function;

6. use the transfer function to write a single differential equation between the output variable and the system input.

This method can illustrated with the following example, where we considered the RLC elctric circuit shown in Fig. 1:

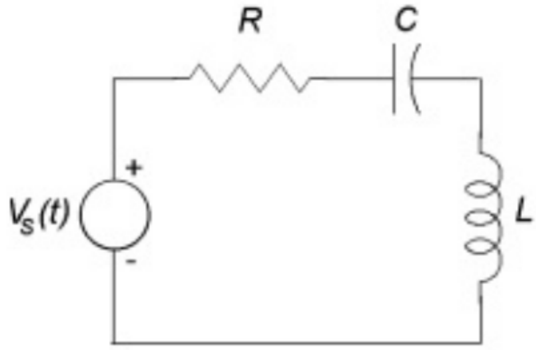

Figure 2.1: RLC circuit

Using Kirchoff's law, we can choose the capacitor voltage $\nu_C(t)$ and the inductor current $i_L(t)$ as state variable, and generates the following pair of state equations:

$$\frac{d}{dt}\begin{bmatrix}\nu_c\\ i_L\end{bmatrix} = \begin{bmatrix}0 & 1/C\\ -1/L & -R/L\end{bmatrix}\begin{bmatrix}\nu_c\\ i_L\end{bmatrix} + \begin{bmatrix}0\\ 1/L\end{bmatrix}V_{in} \tag{2.13}$$

The required output equations is:

$$y(t) = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\nu_c\\ i_L\end{bmatrix} + [0]V_{in} \tag{2.14}$$

Step 1: in Laplace transform form the state equations are:

$$\begin{array}{rcrcrcr} sV_C(s) & = & 0V_C(s) & + & 1/CI_L(s) & + & 0V_s(s)\\ 1/LV_C(s) & = & -1/LV_C(s) & - & R/LI_L(s) & + & 1/LV_s(s)\end{array} \tag{2.15}$$

Step 2: reorganize the state equations:

$$\begin{array}{rcrcr} sV_C & - & 1/CI_L(s) & = & 0V_s\\ 1/LV_C & + & [s+R/L]I_L(s) & = & 1/LV_s\end{array} \tag{2.16}$$

Step 3: in this case we have two simultaneous operational equations in the state variables ν_C and i_L. The output equation requires only ν_C. If Eq. (27.1) is multiplied by $[s+R/L]$,and Eq.(27.2) is multiplied by $1/C$, and the equations added, $I_L(s)$ is eliminated:

$$[s(s+R/L)+1/(LC)]V_C(s) = 1/(LC)V_s(s) \tag{2.17}$$

Step 4: the output equation is $y = \nu_c$. Operate on both sides of Eq. (28) by $[s^2+(R/L)s+1/LC]^{-1}$and write in quotient form:

$$V_C(s) = \frac{1/LC}{s^2+(R/L)s+1/LC}V_s(s) \tag{2.18}$$

Step 5: the transfer function $H(s) = V_C(s)/V_s(s)$ is :

$$H(s) = \frac{1/LC}{s^2+(R/L)s+1/LC} \tag{2.19}$$

Step 6: the differential equation relating $\nu_C$to $V_s$is :

$$\frac{d^2\nu_C}{dt^2} + \frac{R}{L}\frac{d\nu_C}{dt} + \frac{1}{LC}\nu_C = \frac{1}{LC}V_s(t) \tag{2.20}$$

Cramer's Rule, for the solution of set of linear algebric equations , is a useful method to apply to the solution of these equations. In solving for the variable x_i in a set of n linear algbric equations, such as $\mathbf{Ax} = \mathbf{b}$ the rule states:

$$x_i = \frac{\det[\mathbf{A}^{(i)}]}{\det[\mathbf{A}]} \tag{2.21}$$

where $\mathbf{A}^{(i)}$ is another $n \times n$ matrix formed by replacing the *i*th column of $\mathbf{A}$ with the vector $\mathbf{b}$. If:

$$[s\mathbf{I} - \mathbf{A}]\mathbf{X}(s) = \mathbf{B}\mathbf{U}(s) \tag{2.22}$$

then the relationship between the *i*th state variable and the input is:

$$X_i(s) = \frac{\det\left[[s\mathbf{I} - \mathbf{A}]^{(i)}\right]}{\det[s\mathbf{I} - \mathbf{A}]} U(s) \tag{2.23}$$

where $(s\mathbf{I} - \mathbf{A})^{(i)}$is defined to be the matrix formed by replacing the *i*th column of $(s\mathbf{I} - \mathbf{A})$with the column vector $\mathbf{B}$. The differential equation is:

$$\det[s\mathbf{I} - \mathbf{A}]x_i = \det[(s\mathbf{I} - \mathbf{A})^{(i)}]u_k(t) \tag{2.24}$$

Then, riconsidering the previous example, using Cramer's Rule, we can solve for $v_L(t)$ in the following way. The output equation of state model is:

$$v_L = -v_C - Ri_L + V_s(t) \tag{2.25}$$

In the Laplace domain the state equations are:

$$\begin{bmatrix} s & -1/C \\ 1/L & s + R/L \end{bmatrix} \begin{bmatrix} V_c(s) \\ I_L(s) \end{bmatrix} = \begin{bmatrix} V_c(s) \\ I_L(s) \end{bmatrix} V_{in}(s) \tag{2.26}$$

The voltage $V_C(s)$ is given by:

$$V_C(s) = \frac{\det\left[[s\mathbf{I} - \mathbf{A}]^{(1)}\right]}{\det[s\mathbf{I} - \mathbf{A}]} V_{in}(s) = \frac{\det\begin{bmatrix} 0 & -1/C \\ 1/L & (s + R/L) \end{bmatrix}}{\det\begin{bmatrix} s & -1/C \\ 1/L & s + R/L \end{bmatrix}} V_{in}(s) = \frac{1/LC}{s^2 + (R/L)s + (1/LC)} V_{in}(s) \tag{2.27}$$

The current $I_L(s)$ is:

$$I_L(s) = \frac{\det\left[[s\mathbf{I} - \mathbf{A}]^{(2)}\right]}{\det[s\mathbf{I} - \mathbf{A}]} V_{in}(s) = \frac{\det\begin{bmatrix} s & 0 \\ 1/L & 1/L \end{bmatrix}}{\det\begin{bmatrix} s & -1/C \\ 1/L & s + R/L \end{bmatrix}} V_{in}(s) = \frac{R/L}{s^2 + (R/L)s + (1/LC)} V_{in}(s) \tag{2.28}$$

The output equation may be written directly from Laplace transform of Eq. (36) and substituting Eqs.(38-39):

$$\begin{aligned} V_L(s) &= -V_C(s) - RI_L(s) + V_s(s) \\ &= \left[\frac{-1/LC}{s^2 + (R/L)s + (1/LC)} + \frac{-R/L}{s^2 + (R/L)s + (1/LC)} + 1\right] V_s(s) \\ &= \frac{(-1/LC - (R/L)s + (s^2 + (R/L)s + (1/LC))}{s^2 + (R/L)s + (1/LC)} V_s(s) \\ &= \frac{s^2}{s^2 + (R/L)s + (1/LC)} V_s(s) \end{aligned} \tag{2.29}$$

giving the differential equation:

$$\frac{d^2 v_L}{dt^2} + \frac{R}{L}\frac{dv_L}{dt} + \frac{1}{LC} v_L(t) = \frac{d^2 V_s}{dt^2} \tag{2.30}$$

For a single-input single-output (SISO) system the transfer function may be found directly by evaluating the inverse matrix:

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U(s) \tag{2.31}$$

Using the definition of the matrix inverse:

$$[s\mathbf{I} - \mathbf{A}]^{-1} = \frac{adj[s\mathbf{I} - \mathbf{A}]}{det[s\mathbf{I} - \mathbf{A}]}, \tag{2.32}$$

$$\mathbf{X}(s) = \frac{adj[s\mathbf{I} - \mathbf{A}]\mathbf{B}}{det[s\mathbf{I} - \mathbf{A}]}U(s). \tag{2.33}$$

and substituting into the output equation gives:

$$Y(s) \quad = \quad C[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}U(s) + \mathbf{D}U(s) = \left[C[s\mathbf{I} - \mathbf{A}]^{-1} + \mathbf{B} + \mathbf{D}\right]U(s). \tag{2.34}$$

Expanding the inverse in terms of the determinant and the adjoint matrix yields:

$$Y(s) = \frac{Cadj(s\mathbf{I} - \mathbf{A})\mathbf{B} + det[s\mathbf{I} - \mathbf{A}]\mathbf{D}}{det[s\mathbf{I} - \mathbf{A}]}U(s) = H(s)U(s) \tag{2.35}$$

so that the required differential equation may be found by expanding:

$$det[s\mathbf{I} - \mathbf{A}]Y(s) = [Cadj(s\mathbf{I} - \mathbf{A})\mathbf{B} + det[s\mathbf{I} - \mathbf{A}]\mathbf{D}]\,U(s) \tag{2.36}$$

and taking the inverse Laplace transform of both sides.

Chapter 3

Transformation from Classical Form to State-Space Representation

The block diagram provides a convenient method for deriving a set of state equations for a system that is specified in terms of a single input/output differential equation. A set of n state variables can be identified as the outputs of integrators int the diagram, and state equations can be written from the conditions at the inputs to the integrator blocks (the derivative of state variables).

Let the differential equation representing the system be of order n, and without loss of generality assume that the order of the polunomia operators on both sides is the same:

$$(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0)Y(s) = (b_n s^n + b_{n-1} s^{n-1} + \cdots + b_0)U(s) \tag{3.1}$$

We may multiply both sides of the equation by s^{-n} to ensure that all differential operators have been eliminated:

$$\begin{aligned}&(a_n + a_{n-1}s^{-1} + \cdots + a_1 s^{-(n-1)} + a_0 s^{-n})Y(s))\\ &= (b_n + b_{n-1}s^{-1} + \cdots + b_1 s^{-(n-1)} + \cdots + b_0 s^{-n})U(s)\end{aligned} \tag{3.2}$$

from which the output may be specified in terms of a transfer function. If we define a dummy variable $Z(s)$, and split Eq. (49) into two parts:

$$Z(s) = \frac{1}{a_n + a_{n-1}s^{-1} + \cdots + a_1 s^{-(n-1)} + a_0 s^{-n}} U(s) \tag{3.3}$$

$$Y(s) = (b_n + b_{n-1}s^{-1} + \cdots + b_1 s^{-(n-1)} + b_0 s^{-n})Z(s) \tag{3.4}$$

Eq. (30) may be solved for $U(s)$,

$$U(s) = (a_n + a_{n-1}s^{-1} + \cdots + a_1 s^{-(n-1)} + a_0 s^{-n})X(s) \tag{3.5}$$

and rearranged to generate a *feedback* structure that can be used as the basis for a block diagramm:

$$Z(s) = \frac{1}{a_n}U(s) - (\frac{a_{n-1}}{a_n}\frac{1}{s} + \cdots + \frac{a_1}{a_n}\frac{1}{s^{n-1}} + \frac{a_0}{a_n}\frac{1}{s^{-n}})Z(s) \tag{3.6}$$

The dummy variable $Z(s)$ is specified in terms of the system input $u(t)$ and a weighted sum of successive integrations of itself. Equation (51) serves to combine the outputs from the integrators into output $y(t)$.

A set of state equations may be found from the block diagram by assigning the state variables $x_i(t)$ to the outputs of the n integrators. Because of the direct cascade connection of the integrators, the state equations take a very simple form. By inspection:

$$\begin{array}{lcl} \dot{x_1} & = & x_2 \\ \dot{x_2} & = & x_3 \\ \vdots & & \vdots \\ \dot{x}_{n-1} & = & x_n \\ \dot{x}_n & = & -\frac{a_0}{a_n}x_1 - \frac{a_1}{a_n}x_2 - \cdots - \frac{a_{n-1}}{a_n}x_n + \frac{1}{a_n}u(t) \end{array} \tag{3.7}$$

In the matrix form these equations are:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-2} \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0/a_n & -a_1/a_n & \cdots & -a_{n-2}/a_n & -a_{n-2}/a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1/a_n \end{bmatrix} u(t) \tag{3.8}$$

The matrix **A** has a very distinctive form. Each row, except the bottom one, is filled of zeros except for one in the position just above the leading diagonal. Equation (35) is a common form of the state equations, used in control system theory and known as the *phase variable* or companion form. This form leads to a set of state variables which may not correspond to any physical variables within the system.

The corresponding output relationship is specified by Eq.(51) by noting $X_i(s) = Z(s)/s^{(n+1+i)}$.

$$y(t) = b_0x_1 + b_1x_2 + b_2x_3 + \cdots + b_{n-1}x_n + b_nz(t) \tag{3.9}$$

But $z(t) = dx_n/dt$, which is found from the nth state equation in Eq. (34). When substituted into Eq.(36) the output equation is:

$$Y(s) = \left[\left(b_1 - \frac{b_na_0}{a_n}\right) \left(b_1 - \frac{b_na_1}{a_n}\right) \cdots \left(b_1 - \frac{b_na_{n-1}}{a_n}\right) \right] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \frac{b_n}{a_n}u(t) \tag{3.10}$$

Chapter 4

The Matrix Transfer Function

For a multiple-input multiple-output system Eq.(22) is written in ters of the r component input vector $\mathbf{U}(s)$:

$$\mathbf{X}(s) = [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\mathbf{U}(s) \tag{4.1}$$

generating a set of n simultaneous linear equations, where the matrix is $\mathbf{B}$is $n \times r$. The m component system output vector $\mathbf{Y}(s)$ may be found by substituting this solution for $\mathbf{X}(s)$ into the output equation as in Eq.(35):

$$\mathbf{Y}(s) = \mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\{\mathbf{U}(s)\} + \mathbf{D}\{\mathbf{U}(s)\} = \left[\mathbf{C}[s\mathbf{U} - \mathbf{A}]^{-1}B + \mathbf{D}\right]\{\mathbf{U}(s)\} \tag{4.2}$$

and expanding the inverse in terms of the determinant and the adjoint matrix:

$$\mathbf{Y}(s) = \frac{\mathbf{C}\mathrm{adj}(s\mathbf{I} - \mathbf{A})\mathbf{B} + \det[s\mathbf{I} - \mathbf{A}]\mathbf{D}}{\det[s\mathbf{I} - \mathbf{A}]}\mathbf{U}(s) = \mathbf{H}(s)\mathbf{U}(s) \tag{4.3}$$

where $\mathbf{H}(s)$ is defined to be the matrix transfer function realting the output vector $\mathbf{Y}(s)$ to the input vector $\mathbf{U}(s)$:

$$H(s) = \frac{\mathbf{C}\mathrm{adj}(s\mathbf{I} - \mathbf{A})\mathbf{B} + \det[s\mathbf{I} - \mathbf{A}]\mathbf{D}}{\det[s\mathbf{I} - \mathbf{A}]} \tag{4.4}$$

For a system with r inputs $U_1(s), \ldots, U_r(s)$ and m outputs $Y_1(s), \ldots, Y_m(s)$, $\mathbf{H}(s)$ is as $m \times r$ matrix whose elements are indiviudal scalr transfer functions realting a given component of the output $\mathbf{Y}(s)$ to a component of the input $\mathbf{U}(s)$. Expansion of Eq. (41) generates a set of equations:

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \\ \vdots \\ Y_m(s) \end{bmatrix} = \begin{bmatrix} H_{11}(s) & H_{12}(s) & \cdots & H_{1r}(s) \\ H_{21}(s) & H_{22}(s) & \cdots & H_{2r}(s) \\ \vdots & \vdots & \ddots & \vdots \\ H_{m1}(s) & H_{m2}(s) & \cdots & H_{mr}(s) \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \\ \vdots \\ U_r(s) \end{bmatrix} \tag{4.5}$$

where the ith component of the output vector $\mathbf{Y}(s)$ is:

$$Y_i(s) = H_{i1}(s)U_1(s) + H_{i2}(s)U_2(s) + \cdots + H_{ir}(s)U_s(s) \tag{4.6}$$

The elemental transfer function $H_{ij}(s)$ is the scalar transfer function between the ith output component and the jth input component. Equation (61) shows that all the $H_{ij}(s)$ transfer functions in $\mathbf{H}(s)$ have the same denominator factor $\det[s\mathbf{I} - \mathbf{A}]$, giving the important polynomial, or alternatively have the same coefficients on the left-hand side.
If the system has single-input and single-output, $\mathbf{H}(s)$ is a scalr, and the procedure generates the input/output transfer operator directly.

Chapter 5

Frequency Response

5.1 The Concept of Frequency Response

In the steady state, sinusoidal inputs to a linear system generate sinusoidal responses of the same frequency. Even though these responses have the same frequency of the input, they differ in amplitude and phase angle from the input. These differences are functions of frequency.
Before defining frequency response, let us look at a convenient representation of sinusoids. Sinuoids can be represebted as complex number is the amplitude of the sinusoid, and the angle of the complex number is the phase angle of the sinusoids. Thus, $M_1 \cos(\omega t + \phi_1)$can be represented as $M_1 \angle \phi_1$where the frequency, ω, is implicit.
Since a system causes both the amplitude and phase angle of the input to be changed, we can think of the system itself as represented by a complex number, defined so that the product of the input phasor and the system function yields the phasor representation of the output.
Consider the mechanical system in Fig.(2) . If the input force, $F(t)$, is sinusoidal, the steady-state output respose, $x(t)$, of the system is also sinusoidal and at the same frequencxy as the input.

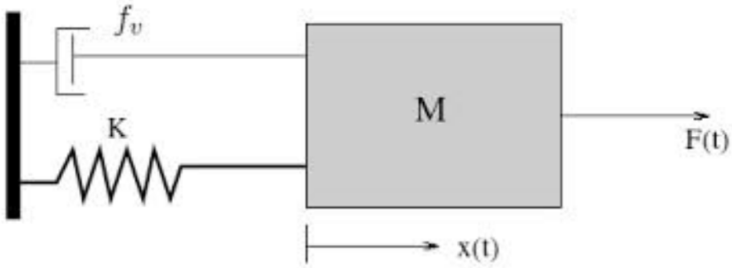

Figure 5.1: System

Assume that the system is represented by the complex number, $M(\omega)\angle\phi(\omega)$. The ouput steady-state sinusoid is found by multiplying the complex number representation of the input by the complex number representation of the system. Thus, the steady-state output sinusoid is:

$$M_0(\omega)\angle\phi_o = M_i(\omega)M(\omega)\angle[\phi_i(\omega) - \phi(\omega)] \tag{5.1}$$

and:

$$\phi(\omega) = \phi_o(\omega) - \phi_i(\omega) \tag{5.2}$$

Equations (65) and (64) form our definition of frequency response. We call $M(\omega)$ the *magnitude frequency response* and $\phi(\omega)$ the *phase frequency response*. The combination of the magnitude and phase frequency responses is called the *frequency response* and is $M(\omega)\angle\phi$.
In other words, we define the magnitude frequency response to be the ratio of the output sinusoid's magnitude to the input sinusoid's magnitude. We define the phase response to be the difference in phase angle between the output and the input sinusoids. Both responses are function of frequency and apply only to the steady-state sinusoidal response of the system.

5.2 Analytical Expressions for the Frequency Response.

Now that we have defined frequency response, let us obtain the analytical expression for it. Consider a system rappresented by a transfer function $G(s)$, with the Laplace transform of general sinusoid, $r(t) = A\cos(\omega t) + B\sin(\omega t) = \sqrt{A^2+B^2}\cos[\omega t - \arctan(B/A)]$as the input. We can represent the input as a phasor in three ways:

1. polar form:,$M_i\angle\phi_i$, where $M_i = \sqrt{A^2+B^2}$and $\phi_i = -\arctan(B/A)$;
2. in rectangular form, $A - jB$;
3. using Euler's formula: $M_i \exp(j\phi_i)$.

We now solve for the forced response portion of $C(s)$, from whic we evaluate the frequency response. Rembering that the Laplace transfor of a sinuoid is:

$$\mathcal{L}[\sin(\omega t)] = \frac{As + B\omega}{s^2+\omega^2} \tag{5.3}$$

we have:

$$C(s) = \frac{As + B\omega}{s^2+\omega^2} G(s) \tag{5.4}$$

We separate the forced solution from the transient solution by performing a partial fraction on Eq.(67). Thus,

$$C(s) = \frac{As + B\omega}{(s+\omega)(s-\omega)} G(s) = \frac{K_1}{s+j\omega} + \frac{K_2}{s-j\omega} + \dots \tag{5.5}$$

where:

$$\begin{aligned} K_1 &= \tfrac{As+B\omega}{s-j\omega} G(s)\,|_{s\longrightarrow -j\omega} = \tfrac{1}{2}(A+jB)G(-j\omega) = \tfrac{1}{2}M_i \exp(-j\phi_i) M \exp(-j\phi_G) = \\ &= \tfrac{M_i M}{2}\exp(j(\phi_i+\phi_G)) \end{aligned} \tag{5.6}$$

$$\begin{aligned} K_2 &= \tfrac{As+B\omega}{s-j\omega} G(s)\,|_{s\longrightarrow +j\omega} = \tfrac{1}{2}(A+jB)G(j\omega) = \tfrac{1}{2}M_i \exp(j\phi_i) M \exp(j\phi_G) = \\ &= \tfrac{M_i M}{2}\exp(j(\phi_i+\phi_G)) = K_1^* \end{aligned} \tag{5.7}$$

For Eqs.(69-70), $K_1^*$is the complex conjugate of K_1,

$$M_G = |G(j\omega)| \tag{5.8}$$

and:

$$\phi_G = \angle G(j\omega) \tag{5.9}$$

The steady-state response is that portion of the partial-fraction expansion that comes from input waveform's poles, or just the first two terms of Eq.(68). Hence, the sinusoidal steady-state output, $C_{ss}(s)$,is:

$$C_{ss}(s) = \frac{K_1}{s+j\omega} + \frac{K_2}{s-j\omega} \tag{5.10}$$

Substituing Eqs. (69-70) into Eq.(73), we obtain:

$$C_{ss}(s) = \frac{\frac{M_i M}{2}\exp(j(\phi_i + \phi_G))_1}{s + j\omega} + \frac{\frac{M_i M}{2}\exp(j(\phi_i + \phi_G))}{s - j\omega} \tag{5.11}$$

Taking the inverse Laplace transformation, we obtain:

$$\begin{aligned} c(t) &= M_i M_G \left(\frac{\exp(-j(\omega t + \phi_i + \phi_G)) + \exp(j(\omega t + \phi_i + \phi_G))}{2}\right) \\ &= M_i M_G \cos(\omega t + \phi_i + \phi_G) \end{aligned} \tag{5.12}$$

which can be rapresented in phasor formas $M_o \angle \phi_o = (M_1 \angle \phi_i)(M_G \angle \phi_G)$, where $M_G \angle \phi_G$ is the frequency response function. But, from Eqs.(71) and (72) $M_G \angle \phi_G$. In other words, the frequency response of a system whose transfer function is $G(s)$ is:

$$G(j\omega) = G(s)|_{s \longrightarrow j\omega} \tag{5.13}$$