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Aerospace System Guidance and Control
Lesson II

Dynamics and Kinematics

POLITECNICO DI MILANO

DIPARTIMENTO DI
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Abstract
When thinking about any kind of vehicle the first question that comes to mind is "how does it moves?". In order to build a vehicle it is of the utmost importance to be able to understand the mechanism that permits it to move. This lesson is about the methods we have to understand the motion of a vehicle in three dimensions and how we can predict them. Knowledge about dynamics is important to design and control any kind of vehicle.We will refer to an aerospace vehicle that is, in general, able to move in three dimensions and rotate in any kind of configuration.

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Chapter 1

Reference Frames

Reference frames are necessary to address the dynamics and the kinematics of an object or vehicle. To understand the importance of a reference frame just think about the "position" of an object. "The pen lies 10 cm on the right of the book", such a statement is useless unless you define a direction, a verse and a magnitude: a "position" vector. We can express one vector in terms of other vectors, for example "the pen is 6 cm along one side of the book and 8 cm on along the other side of the book". Mathematically speaking we must use unitary vectors to construct the base of the vector space that we are using to define the position of the pen. In the example we have used two sides of a book that are, of course, orthogonal: we have used an orthogonal reference frame. It is possible to use different kinds of unitary vectors groups, but the most simple reference frame has orthogonal unitary vectors. We will use a right-hand reference frame so that you can easily picture it using your right hand: the first axis is the thumb, the second axis is the index and the third one is the middle finger.

1.1 Inertial Reference Frame

Going by Newton's definition an inertial frame is a reference frame that is still or is moving at constant velocity. In reality there are no inertial systems, but depending on the case we can assume that a certain reference can be considered inertial even if it is accelerating or spinning. If we walk along a corridor in the university for 3 minutes we can assume that a reference frame fixed with the Earth is inertial, even if we know that the planet is spinning around its axis and around the sun and so on. However the rotation of the planet is so small that in few minutes the difference won't be noticeable.

1.2 NED Reference Frame

In the aeronautics one of the most used reference frame is the North East Down frame. This means that the first axis points to the geographical north, the second to the geographical east and the third one towards the ground. In many application it can be considered inertial, however for a intercontinental plane this might not be the case.

1.3 Body Fixed Reference Frame

This reference frame is attached to a body that accelerates and spin, therefore it is a non-inertial reference frame. A special reference frame of this kind is the so called Principal Axis of Inertia reference frame, whose axis are coincident with the principal axis of inertia of a body. This will be useful later on when we will speak about the rotation dynamics.

1.4 Coordinate Systems

Even if the reference frame is a tree-orthogonal-axis frame we can use different coordinates to define the position of a point. The easiest to use is the rectangular set where we have x y and z that are the distance from the origin of a the projection of a point along that axis. in a two dimensional case we can use polar coordinates: the distance from the origin r and the anomaly ϑ that is the angle between the vector radius and one reference axis. Adding one dimension we have cylindrical coordinates that adds the z axis similarly to the rectangular case and the spherical coordinates where another angle ϕ defines another inclination w.r.t another reference axis or plane. In general Rectangular coordinates are preferred since the law of dynamics can be expressed quite easily, however this is problem dependent.

Chapter 2

Law of Dynamics

2.1 Center of Mass

The gravity force is one of the most important forces acting on an aerospace vehicle, being it an airplane or a spacecraft. Gravity is strictly correlated to the concept of mass, but we will not address matter of physics or philosophy and we will concentrate onto more practical aspects. This means that we must understand how acts the gravity force and how we can model it. The center of mass is that point in (or outside, depending on circumstances) the body where we can consider the gravity force to be applied. To determine the center of mass of a body constituted by N smaller parts we can use the following:

$$x_{cg} = \frac{\sum_i^N x_i m_i}{\sum_i^N m_i}$$

Of course when considering a continuous body we should use an integral relation, however for practical purposes the center of mass of a vehicle can be obtained by dividing it into sub parts. In general if the geometry of the body or one of its part is known and the density of the constituting material is constant in all the body we have that the center of mass coincides with the centroid of the body, that is a geometrical property. This implies that for complex geometries it is possible to split the body into simpler parts and then compute the center of mass.
This important location will be useful later on also for other considerations.

2.2 Linear Momentum Conservation

2.2.1 Inertial Reference Frame

The first law of dynamics, in a inertial reference frame, assumes the following expression

$$\frac{d}{dt}(m\mathbf{v}) = \mathbf{f}_{ext}$$

where $\mathbf{v}$ is the velocity of the body, m its mass and $\mathbf{f}_{ext}$ the equivalent external force applied to the body. From this we can derive an Ordinary Differential Equation (ODE) to get how the velocity of the body changes in time:

$$\dot{\mathbf{v}} = \frac{1}{m}(\mathbf{f}_{ext} - \dot{m}\mathbf{v})$$

then the position can be easily computed integrating directly the velocity

$$\dot{\mathbf{r}} = \mathbf{v}$$

2.2.2 Non-inertial reference frame

If we write the laws of dynamics in a non inertial reference frame rotating at constant speed we would have

$$
\begin{aligned}
\dot{\mathbf{r}}_i &= \dot{\mathbf{r}} + ! \times \mathbf{r} \\
\dot{\mathbf{v}}_i &= \dot{\mathbf{v}} + 2! \times \mathbf{v} + ! \times ! \times \mathbf{r}
\end{aligned}
$$

where $\mathbf{v}_i$ and $\mathbf{r}_i$ are the position and velocity vector in the inertial reference frame. Substituting the expressions that we saw before we would have

$$
\begin{aligned}
\mathbf{v}_i &= \dot{\mathbf{r}} + ! \times \mathbf{r} \\
\frac{1}{m}\left(\mathbf{f}_{ext} - \dot{m}\mathbf{v}_i\right) &= \dot{\mathbf{v}} + 2! \times \mathbf{v} + ! \times ! \times \mathbf{r}
\end{aligned}
$$

therefore

$$
\begin{aligned}
\dot{\mathbf{r}} &= \mathbf{v} \\
\dot{\mathbf{v}} &= \frac{1}{m}\left(\mathbf{f}_{ext} - \dot{m}\mathbf{v}\right) - 2! \times \mathbf{v} - ! \times ! \times \mathbf{r}
\end{aligned}
$$

This set of equations is valid for a constantly rotating body, however if the angular velocity is changing there is another term to consider. In many practical situation this complicated modification is avoided and just the instantaneous angular velocity is used.

2.2.3 Forces

In the term $\mathbf{f}_{ext}$ we list the sum of all forces that are applied to the body in its center of mass. This forces can be of different nature

$$\mathbf{f}_{ext} = \sum \mathbf{f}_{aerodynamics} + \sum \mathbf{f}_{propulsive} + \sum \mathbf{f}_{gravitational} + \sum \mathbf{f}_{electromagnetic} + \dots$$

we will limit it to the main components:

$$\mathbf{f}_{ext} = \mathbf{t} + \mathbf{f}_a + m\mathbf{g}$$

where $\mathbf{t}$ is the thrust, $\mathbf{f}_a$ the aerodynamic forces and $\mathbf{g}$ is the gravity force per unit mass (acceleration). $\mathbf{t}$ usually depends on the engine of the vehicle: it can be a propeller, a jet engine or even a rocket motor. The nature of the engine will influence the expression of the thrust, since it depends on environmental quantities such as the air density (or even its absence). It is easy to express in body fixed frame.
The aerodynamic forces can be written as

$$\mathbf{f}_a = \frac{1}{2}\rho v^2 S \begin{Bmatrix} C_x \\ C_y \\ C_z \end{Bmatrix}$$

where ρ is the air density, v is the modulus of the speed relative to the air (i.e. vehicle speed and wind speed sums together), S is a reference surface and Cs are the coefficients obtained by experimentation or computation. Usually such coefficients are computed by measuring the force the air puts on a body and dividing it by the so called dynamic pressure $\frac{1}{2}\rho v^2$ and the reference surface S. These coefficient depends on many parameters such as the Mach Number, the Reynolds number, the fluid orientation, the shape of the body and so on.
$\mathbf{g}$ is measured as an acceleration and depends on the distance from the center of the planet with a law such as $g \propto r^{-2}$, therefore it weakens the more we distance ourselves from the center of the Earth. For near surface operation it can be considered quite constant.

Attention must be paid on the reference frames where the forces are expressed: for example if the gravity force is computed in the inertial reference frame and the equation of motion are expressed in the principal axis of inertia reference frame it is needed a rotation to orient the force in the used reference frame. Rotation of a vector in 3D space is usually obtained by multiplying the vector with a direction cosine matrix that changes only its orientation and not its modulus.

2.3 Angular Momentum Conservation

2.3.1 NED reference frame

Computing the angular momentum conservation in the NED reference frame, as in many others frames, is quite difficult since the inertia tensor expressed in such frames is not constant but depends inherently on the orientation. To grasp this concept let us take a cylinder whose longitudinal axis is z. The inertia along this axis is way inferior with respect to the others, therefore with the same torque applied it will rotate at much higher speed on the z axis. If we take an inertial reference frame with the z axis vertical and we impose the two axis to be parallel, then the inertia tensor will be the same. However if we let the cylinder rotate so that its z axis is aligned with the horizontal plane, the inertia expressed in the inertial frame has changed, while the inertia in the body frame has not.

2.3.2 Principal Inertia Axis reference frame

We can express the rotation dynamics in the principal axis of inertia in order to simplify the problem. In this reference the inertia tensor is kept constant (if mass doesn't change) and it is easier to handle the computation since the tensor is simplified. By definition the center of this frame is the center of mass, therefore under these hypothesis we can easily write

$$\mathbf{I}\frac{d}{dt}\omega = \omega \times \mathbf{I}\omega + \mathbf{m}$$

where the inertia tensor assumes the particular form of a diagonal tensor.

$$\mathbf{I} = \begin{bmatrix} \mathbf{I}_x & 0 & 0 \\ 0 & \mathbf{I}_y & 0 \\ 0 & 0 & \mathbf{I}_z \end{bmatrix}$$

The principal axis of inertia of a body can be computed knowing the inertia tensor in any kind of reference frame that is centered on the center of mass and fixed with the body. Simply by applying the diagonalization procedure we can determine the rotation that allows the matrix representing the inertia tensor to be diagonal.

Chapter 3

Kinematics

With the term Kinematics we will simply refer to the variation in time of the attitude of a body in space. From the dynamics we can compute the angular velocity history, however we need to take another step to understand how a body rotates in space given its angular velocity.

3.1 Direction Cosine Matrix

The direction cosine matrix is a three by three matrix that represent the Cartesian components of a reference frame in another reference frame. we define such matrix as $\mathbf{R}_{12}$ such that holds the following relation

$$\mathbf{x}_1 = \mathbf{R}_{12}\mathbf{x}_2$$

where $\mathbf{x}_1$ is a vector in a reference frame 1 and $\mathbf{x}_2$ is the same vector expressed in the frame 2. The matrix $\mathbf{R}_{12}$ is an orthogonal matrix with determinant equal to 1 that permits the rotation from the reference 2 to the reference 1. Its column are the coordinates of each axis of frame 2 written in components of the frame 1. These components can be expressed as cosines of angles between axis, hence the name.
The variation in time of a Direction Cosine Matrix can be expressed as

$$\frac{d}{dt}\mathbf{R}_{12} = -\begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix}\mathbf{R}_{12}$$

however we must enforce orthogonality quite often, if we do not want numerical errors to add up and distort also other measurement. A simple, yet costly, procedure to enforce orthogonality can be

$$\mathbf{R}'_{12} = \frac{3}{2}\mathbf{R}_{12} - \frac{1}{2}\mathbf{R}_{12}\mathbf{R}_{12}^T\mathbf{R}_{12}$$

3.2 Euler Angles

Euler angles, in general, are the most natural parametrization of attitude, since it is easier to assign them a physical and visible meaning. Basically the three angles represent three consecutive rotations around three axis. Since the rotation are in sequence the second rotation is around an axis that is not the axis of the initial reference frame but it is around an intermediate axis. The DCM can be expressed as a multiplication of three different DCM, each one representing a rotation around one axis, where the first rotation is on right.

$$\mathbf{R}_{12} = \mathbf{R}_{1g}\mathbf{R}_{gi}\mathbf{R}_{i2}$$

The DCM of a rotation around one axis can be

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta & \sin\vartheta \\ 0 & -\sin\vartheta & \cos\vartheta \end{bmatrix}$$

The variation in time of such parameters is not trivial and depends on the sequence of rotation. Here we report the differential equation for the combination of rotations $z - y - x$.

$$\begin{cases} \dot{\varphi} & = \frac{r\cos\psi + q\sin\psi}{\cos\vartheta} \\ \dot{\vartheta} & = q\cos\psi - r\sin\psi \\ \dot{\psi} & = p + (q\sin\psi + r\cos\psi)\tan\vartheta \end{cases}$$

There is, however, a problem with Euler angles so that normal on-board computers for attitude do not implement such parameters. This problem, often called gymbal-lock, lies when one of the angles approach a particular value and the derivative of one angles goes to infinity: there are singularities. In the example above we can see that if ϑ reaches $90°$ $\dot{\varphi}\&\dot{\psi} \to \infty$ and the solution diverges.

If the motion is well confined it is possible to use Euler angles without problems, but in other cases when the body can change significantly its attitude the problem of singularities cannot be neglected. As will be clearer in the following lessons, having to use trigonometric functions on an on-board controller routine is always avoided if possible.

3.3 Quaternions

Quaternions are the solution to singularities in attitude parametrization, however they need 4 scalar components and quite a complex algebra behind. We will not enter much in the detail of quaternion math, but will simply address what is useful for us.

$$\mathbf{q} = \begin{Bmatrix} x\sin\vartheta \\ y\sin\vartheta \\ z\sin\vartheta \\ \cos\vartheta \end{Bmatrix}$$

where ϑ is the Euler angle[1] and x, y, z the components of the Euler axis. They do represent a rotation from a reference frame to another reference frame, using vectors with four components. It is possible to trace back a quaternion into a DCM and rotate a 3D vector in space.

$$\mathbf{R} = \begin{bmatrix} q_1^2 - q_2^2 - q_3^2 + q_4^2 & 2(q_1q_2 + q_3q_4) & 2(q_1q_3 - q_2q_4) \\ 2(q_1q_2 - q_3q_4) & -q_1^2 + q_2^2 - q_3^2 + q_4^2 & 2(q_2q_3 + q_1q_4) \\ 2(q_1q_3 + q_2q_4) & 2(q_2q_3 - q_1q_4) & -q_1^2 - q_2^2 + q_3^2 + q_4^2 \end{bmatrix}$$

The variation in time of a quaternion can be computed as

$$\dot{\mathbf{q}} = \frac{1}{2}\Omega\mathbf{q}$$

where

$$\Omega = \begin{bmatrix} 0 & r & -q & p \\ -r & 0 & p & q \\ q & -p & 0 & r \\ -p & -q & -r & 0 \end{bmatrix}$$

where p, q, r are the the scalar components of the angular velocity vector.

[1] another kind of parametrization that involves the rotation of one single angle around one single axis, however there is no rule for consecutive rotation, hence useless for our approach.

Chapter 4

Linear Systems and State Space representation

Every dynamical system can be represented using differential equation, however in case of linear system there is a special representation of such equations that can be used in a very powerful way. Should be noticed that no real system is not even remotely linear, however there are numerous cases when it is possible to approximate the system representation and use a linear model.
To understand the process let us take the angular momentum conservation,written in body axis and under the hypothesis of constant mass and inertia.

$$\mathbf{I}\frac{d}{dt}\omega = \omega \times \mathbf{I}\omega + \mathbf{m}$$

now if we assume that the nominal angular velocity of the body is null the non linear term $\omega \times \mathbf{I}\omega$ is made by the product of two angular velocity components that should be null. Even if the velocity is not null we can at least assume that it will be very small: the product of two small quantities is even smaller and thus negligible. Of course a proper procedure would require to compare the order of magnitude of all the equation components. If we neglect such term the equation becomes linear

$$\mathbf{I}\frac{d}{dt}\omega = \mathbf{m}$$

however this is not enough and we need to reconstruct the attitude kinematics. All the presented procedure for attitude propagation are non linear, however we can make other assumptions and reduce the problem even more. Let us represent the kinematics through Euler angles.

$$\begin{cases} \dot{\varphi} & = \frac{r\cos\psi + q\sin\psi}{\cos\vartheta} \\ \dot{\vartheta} & = q\cos\psi - r\sin\psi \\ \dot{\psi} & = p + (q\sin\psi + r\cos\psi)\tan\vartheta \end{cases}$$

Then let us assume that the angles describing the attitude should be small, less than 20° for example, for all the motion that we are interested in. We can make the model linear by series expansion. To make it simple

$$\begin{aligned} \sin x &\simeq x \\ \cos x &\simeq 1 \\ \tan x &\simeq x \end{aligned}$$

therefore

$$\begin{cases} \dot{\varphi} & = r + q\psi \\ \dot{\vartheta} & = q - r\psi \\ \dot{\psi} & = p + (q\psi + r)\vartheta \end{cases}$$

now we assumed that both angular velocities and angles are small, therefore the products such as $q\psi$ are negligible. The conclusion is

$$\begin{cases} \dot{\varphi} & = r \\ \dot{\vartheta} & = q \\ \dot{\psi} & = p \end{cases}$$

substituting in the previous equation we have

$$\begin{cases} I_x \frac{d^2}{dt^2}\psi & = m_x \\ I_y \frac{d^2}{dt^2}\vartheta & = m_y \\ I_z \frac{d^2}{dt^2}\varphi & = m_z \end{cases}$$

This second order system is now linear. We can take a step further and re-arrange it in a matrix form

$$\frac{d}{dt}\begin{Bmatrix} \dot{\psi} \\ \dot{\vartheta} \\ \dot{\varphi} \\ \psi \\ \vartheta \\ \varphi \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{\psi} \\ \dot{\vartheta} \\ \dot{\varphi} \\ \psi \\ \vartheta \\ \varphi \end{Bmatrix} + \begin{bmatrix} 1/I_x & 0 & 0 \\ 0 & 1/I_y & 0 \\ 0 & 0 & 1/I_z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} m_x \\ m_y \\ m_z \end{Bmatrix}$$

in a more compact and general form we can write

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

In the future lessons this form will show you all its power by simplifying the design of the whole GNC system. The name of this representation is usually "state space" representation, where the state is the vector $\mathbf{x}$ that contains all the "states" of the dynamical system take in consideration. The vector $\mathbf{u}$ is usually linked to the external input of the system, meaning all the factors that are not considered part of the system state. In this category falls bot the control actions, controlled input that are meant to produce a controlled evolution of the system, and external disturbances. The matrix $\mathbf{A}$ tells us what is the natural evolution of the system if no other input is present. The result is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$
$$\mathbf{x}(t = 0) = \mathbf{x}_0$$

where we have put in evidence the initial condition of the system. In our case the form of matrix $\mathbf{A}$ tell us that whichever the initial angular position (within our hypothesis of small angles!) takes the body the attitude evolution in time does not depend by it. On the other hand if we have non null angular velocity we have that the attitude angles change linearly in time with a fixed velocity equal to the initial one. Pay attention that this system with initial velocity non null, sooner or later will go out the boundaries that we have imposed on the model and therefore the linear model will not be representative of the real system behavior. This suggest us that such system with such hypothesis are meant to evolve within some pre-fixed boundaries and that the control action is used to enforce such boundaries.
The previous analysis can be made quite easily using matrix algebra: let us try to compute the eigenvectors of matrix $\mathbf{A}$. It is easy to see that the system has 6 null eigenvalues and this means

that without an external control action the minimum disturbance could make the whole system diverge. Such analysis and much more will be explained later on.

4.1 Blimp model

Let us take our blimp and write the equations of motion for each axis, simplify them and obtain a linear system that describes the whole motion.
First of all we must take into account all the forces and torques that are applied to the vehicle:

- weight, pointing towards the ground;
- lift, pointing skyward;
- thrust, magnitude and direction given by the controller;
- aerodynamic drag, pointing in the direction opposite to the velocity of the blimp relative to the air.

4.1.1 Vertical direction

If the engines are switched off and the blimp is standing still, meaning no drag, we have an equilibrium condition with the lift force equilibrating the weight of the vehicle. You can call the height achieved by the blimp as its equilibrium height: without external action the system will tend to reach that altitude and maintain it. Please notice that the lift, in this case,is generated by the balloons, not by an airfoil or a wing since they require a non null stream velocity not compatible with a standing still blimp.

$$mg = \bar{l}$$

where $\bar{l}$ is the lift force at equilibrium condition. Since the force depends on the external pressure a big change in altitude can make this force change, however if we limit ourselves to a hovering condition with small altitude variation we can assume that the lift force is equal to the weight of the system. This also implies almost no drag. Now, let us add a component of the thrust t_z that is positive upwards,meaning that it produces a positive increment of altitude considering a z-axis pointing to the zenith. The first law of dynamics give us

$$\begin{aligned}
m\dot{v}_z &= \bar{l} - mg + t_z \\
\dot{v}_z &= \frac{\bar{l} - mg}{m} + \frac{t_z}{m} \\
\ddot{z} &= \underbrace{\frac{\bar{l} - mg}{m}}_{0} + \frac{t_z}{m} \\
\ddot{z} &= \frac{t_z}{m}
\end{aligned}$$

in state space representation

$$\frac{d}{dt}\begin{Bmatrix}\dot{z}\\ z\end{Bmatrix} = \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\begin{Bmatrix}\dot{z}\\ z\end{Bmatrix} + \begin{bmatrix}\frac{1}{m}\\ 0\end{bmatrix} t_z$$

4.1.2 Translation

now let us address the translation in the horizontal plane. since the propeller are mounted such that the thrust is always in the vertical plane we can assume that the lateral motion is null. Let us address the motion along the x axis of the blimp, therefore in the non-inertial system. At first we assume that the body is not rolling, therefore we have

$$\begin{aligned} m\dot{v}_x &= t_x - d \\ \dot{v}_x &= \frac{1}{m}\left(t_x - \frac{1}{2}\rho v_x^2 S C_D\right) \\ \ddot{x} &= \frac{1}{m}\left(t_x - \frac{1}{2}\rho S C_D \dot{x}^2\right) \end{aligned}$$

assuming small variation of altitude such that ρ can be considered constant we have

$$\frac{d}{dt}\begin{Bmatrix}\dot{x}\\ x\end{Bmatrix} = \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\begin{Bmatrix}\dot{x}\\ x\end{Bmatrix} + \begin{bmatrix}\frac{1}{m}\\ 0\end{bmatrix} t_x - \begin{bmatrix}\frac{\rho S C_D}{2m}\\ 0\end{bmatrix}\dot{x}^2$$

now we can simplify the non-linear system in two different ways. The most simple requires the direct elimination of the drag since the velocity of the blimp can be considered small. If we want to take into account the little contribution of the drag we can assume that the blimp is moving at constant speed in the nominal configuration and have the drag term linearized. In the latter case we have

$$\frac{d}{dt}\begin{Bmatrix}\dot{x}\\ x\end{Bmatrix} = \begin{bmatrix}-\frac{\rho S C_D \bar{\dot{x}}}{m} & 0\\ 1 & 0\end{bmatrix}\begin{Bmatrix}\dot{x}\\ x\end{Bmatrix} + \begin{bmatrix}\frac{1}{m}\\ 0\end{bmatrix} t_x$$

It is easy to find out that the eigenvalues of the matrix $\mathbf{A}$ in this case are 0 and $-\frac{\rho S C_D}{m}$. This means that the drag has a stabilizing factor, in fact if the velocity increases from the nominal condition the drag will increase and try to reduce it. Remember that this assumptions valid only in a strictly limited case.

4.1.3 Rotation

If the engines are off the blimp should not rotate at all, if the weights on it are well balanced. Moreover we are not interested in having the blimp pitch or roll, therefore there are no actuators that will try to modify such rotations and the simplest thing to do is to assure that the system is stable against perturbations torques applied on such axis.
What we do want to obtain, on the other hand, is a controlled rotation on the yaw axis: if the blimp can only go forward or backward the only way to make it go left or right is to rotate it. The only force that can produce a torque is the propeller torque and the corresponding drag. This kind of drag is not easy to model and it is also very low if the maneuver is slow,therefore we can neglect it and write

$$I_y \dot{q} = m_y$$

where the non linear term is null since the other two angular velocities are null. Of course in a stability analysis on the other two axis we should take into account a non zero yaw velocity. If we limit the yaw angle to be under 20 degrees we can simply write

$$\ddot{\vartheta} = \frac{m_y}{I_y}$$

in the state space representation

$$\frac{d}{dt}\begin{Bmatrix}\dot{\vartheta}\\ \vartheta\end{Bmatrix} = \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\begin{Bmatrix}\dot{\vartheta}\\ \vartheta\end{Bmatrix} + \begin{bmatrix}\frac{1}{I_y}\\ 0\end{bmatrix} m_y$$

Let us take a look back at the Euler angles variation in time assuming, small ψ and not small ϑ

$$\begin{cases}\dot{\varphi} & = \frac{r+q\psi}{\cos\vartheta}\\ \dot{\vartheta} & = q\\ \dot{\psi} & = (q\psi + r)\tan\vartheta\end{cases}$$

even if we assume $r = 0$ there are troubles.

$$\begin{cases}\dot{\varphi} & = \frac{q}{\cos\vartheta}\psi\\ \dot{\vartheta} & = q\\ \dot{\psi} & = q\psi\tan\vartheta\end{cases}$$

let us think to have $\vartheta = 45°$, the system will be

$$\begin{cases}\dot{\varphi} & = \frac{2q}{\sqrt{2}}\psi\\ \dot{\vartheta} & = q\\ \dot{\psi} & = q\psi\end{cases}$$

This suggest that a small pitching perturbation on ψ coupled with a yaw angular velocity can produce a diverging solution for the other two angles, meaning that the attitude could be lost after just one turn. In order to reduce this effect the maneuver should be slow, therefore if we want to have a big turn we should make it slow.

4.1.4 Complete model for level flight

Now let us write the whole state space representation for the vehicle

$$\frac{d}{dt}\begin{Bmatrix}\dot{z}\\ z\\ \dot{x}\\ x\\ \dot{\vartheta}\\ \vartheta\end{Bmatrix} = \begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -\frac{\rho S C_D \dot{x}}{m} & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\end{bmatrix}\begin{Bmatrix}\dot{z}\\ z\\ \dot{x}\\ x\\ \dot{\vartheta}\\ \vartheta\end{Bmatrix} + \begin{bmatrix}\frac{1}{m} & 0 & 0\\ 0 & 0 & 0\\ 0 & \frac{1}{m} & 0\\ 0 & 0 & 0\\ 0 & 0 & \frac{1}{I_y}\\ 0 & 0 & 0\end{bmatrix}\begin{Bmatrix}t_z\\ t_x\\ m_y\end{Bmatrix}$$

however we have not addressed yet the control actions t_x, t_z and m_y. On the vehicle there are two propellers that produce a varying thrust, not dependent in magnitude one from the other, therefore we can say that we have t_{sx} and t_{dx}. The torque m_y is produced by the difference of these two thrust if we consider both of them at a distance b from the center of mass, moreover the propellers are mounted on a bar that can rotate of an angle η.
Therefore the thrust components are

$$\begin{Bmatrix}t_z\\ t_x\\ m_y\end{Bmatrix} = \begin{bmatrix}\sin\eta & \sin\eta\\ \cos\eta & \cos\eta\\ b & -b\end{bmatrix}\begin{Bmatrix}t_{sx}\\ t_{dx}\end{Bmatrix}$$

In this case the system is no more linear or time invariant.

$$\frac{d}{dt}\begin{Bmatrix}\dot{z}\\ z\\ \dot{x}\\ x\\ \dot{\vartheta}\\ \vartheta\end{Bmatrix} = \begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -\frac{\rho S C_D \dot{x}}{m} & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\end{bmatrix}\begin{Bmatrix}\dot{z}\\ z\\ \dot{x}\\ x\\ \dot{\vartheta}\\ \vartheta\end{Bmatrix} + \begin{bmatrix}\frac{\sin\eta}{m} & \frac{\sin\eta}{m}\\ 0 & 0\\ \frac{\cos\eta}{m} & \frac{\cos\eta}{m}\\ 0 & 0\\ \frac{b}{I_y} & -\frac{b}{I_y}\\ 0 & 0\end{bmatrix}\begin{Bmatrix}t_{sx}\\ t_{dx}\end{Bmatrix}$$

However we have more than one control variable, therefore we can also decide to have t_{sx} and t_{dx} not independent. If we enforce that the thrusting produced by the propellers is constant we have an interesting development

$$\begin{aligned}\bar{t} &= t_{sx} + t_{dx} \\ t_z &= \bar{t}\sin\eta \simeq \bar{t}\eta \\ t_x &= \bar{t}\cos\eta \simeq \bar{t} \\ m_y &= 2t_{sx} - \bar{t}\end{aligned}$$

therefore

$$\frac{d}{dt}\begin{Bmatrix}\dot{z}\\ z\\ \dot{x}\\ x\\ \dot{\vartheta}\\ \vartheta\end{Bmatrix} = \begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -\frac{\rho S C_D \dot{\bar{x}}}{m} & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\end{bmatrix}\begin{Bmatrix}\dot{z}\\ z\\ \dot{x}\\ x\\ \dot{\vartheta}\\ \vartheta\end{Bmatrix} + \begin{bmatrix}0 & \frac{\bar{t}}{m}\\ 0 & 0\\ 0 & 0\\ 0 & 0\\ \frac{2b}{I_y} & 0\\ 0 & 0\end{bmatrix}\begin{Bmatrix}t_{sx}\\ \eta\end{Bmatrix} + \begin{Bmatrix}0\\ 0\\ \frac{\bar{t}}{m}\\ 0\\ -\frac{b\bar{t}}{I_y}\\ 0\end{Bmatrix}$$

which is a linear biased system. If we want to have the blimp at a fixed altitude and moving at constant speed we can make use of the state space representation in this powerful way:

$$\begin{aligned}\mathbf{e} &= \begin{Bmatrix}\dot{z}-\dot{\bar{z}} & z-\bar{z} & \dot{x}-\dot{\bar{x}} & x-\bar{x}\end{Bmatrix} \\ \mathbf{e} &= \begin{Bmatrix}e_{v_z} & e_z & e_{v_x} & e_x\end{Bmatrix}\end{aligned}$$

where the indicates the nominal condition.

$$\begin{Bmatrix}e_{v_z}\\ e_z\\ e_{v_x}\\ e_x\end{Bmatrix} = \begin{bmatrix}0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & -\frac{\rho S C_D \dot{\bar{x}}}{m} & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{Bmatrix}e_{v_z}\\ e_z\\ e_{v_x}\\ e_x\end{Bmatrix} + \begin{bmatrix}0 & \frac{\bar{t}}{m}\\ 0 & 0\\ 0 & 0\\ 0 & 0\end{bmatrix}\begin{Bmatrix}t_{sx}\\ \eta\end{Bmatrix}$$

In this case we see that just η can change the system behavior and it cannot influence the translation dynamics. This is perfectly normal for the linearized system: having a constant thrusting force means that the system can achieve and maintain equilibrium on its own.

$$\begin{aligned}m\ddot{x} &= \bar{t} - \frac{1}{2}\rho S C_D \bar{v}_x^2 \\ 0 &= \bar{t} - \frac{1}{2}\rho S C_D \bar{v}_x^2 \\ \bar{v}_x^2 &= \frac{\bar{t}}{\frac{1}{2}\rho S C_D} \\ \bar{v}_x &= \sqrt{\frac{\bar{t}}{\frac{1}{2}\rho S C_D}}\end{aligned}$$

therefore

$$\begin{Bmatrix}e_{v_x}\\ e_x\end{Bmatrix} = \begin{bmatrix}-\sqrt{\frac{\rho S C_D 2\bar{t}}{m^2}} & 0\\ 1 & 0\end{bmatrix}\begin{Bmatrix}e_{v_x}\\ e_x\end{Bmatrix}$$

since the sign of the stabilizing term has not changed we can deduce that the system has an eigenvalue equal to $-\sqrt{\frac{\rho S C_D 2\bar{t}}{m^2}}$ that is stabilizing the blimp translational velocity. Of course the other eigenvalue is zero, that suggest the continuous change of position. This is the obvious conclusion of the model where we set the velocity to be constant and non-zero.
If we want to have the blimp fixed in a position we should make other assumptions.

Appendix

Function name	Description	Notes
angle2dcm	pass from Euler angles to DCM	
quat2dcm	pass from quaternion to DCM	scalar is first
dcm2quat	pass from DCM to quaternion	scalar is first
dcm2angle	pass from DCM to Euler angles	
angle2quat	pass from Euler angles to quaternion	scalar is first
quat2angle	pass from quaternion to Euler angles	scalar is first
atmoscoesa	1976 COESA model for atmosphere	
ode45	ODE solver function	needs $f(x)$
ode113	ODE solver function	needs $f(x)$

Type "help namefunction" to have the description of input/output and options of the selected function.